Commun. Combin., Cryptogr. & Computer Sci., 1 (2023), 39–54

ISSN:2783-5456

Communications in Combinatorics, Cryptography & Computer Science

Journal Homepage: http://cccs.sgh.ac.ir

Norms Over Intuitionistic Fuzzy Subgroups on Direct Product of Groups

Rasul Rasuli*

Department of Mathematics, Payame Noor University (PNU), P. O. Box 19395-3697, Tehran, Iran.

Abstract

In this paper, by using norms (t-norm T and s-norm S), the notion of intuitionistic fuzzy subgroups on direct product of groups will be introduced. Also intersection and normality of them will be defined and investigated some properties of them. Finally, some results of them under group homomorphisms will be provided.

Keywords: Group theory, theory of fuzzy sets, intuitionistic fuzzy groups, norms, homomorphisms, intersection.

2020 MSC: 20A15, 03E72, 47A30, 20K30, 13C40.

1. Introduction

In mathematics and abstract algebra, group theory studies the algebraic structures known as groups. The concept of a group is central to abstract algebra: other well-known algebraic structures, such as rings, fields, and vector spaces, can all be seen as groups endowed with additional operations and axioms. Groups recur throughout mathematics, and the methods of group theory have influenced many parts of algebra. Linear algebraic groups and Lie groups are two branches of group theory that have experienced advances and have become subject areas in their own right. In classical set theory, the membership of elements in a set is assessed in binary terms according to a bivalent condition an element either belongs or does not belong to the set. By contrast, fuzzy set theory permits the gradual assessment of the membership of elements in a set; this is described with the aid of a membership function valued in the real unit interval [0, 1]. Fuzzy sets generalize classical sets, since the indicator functions (aka characteristic functions) of classical sets are special cases of the membership functions of fuzzy sets, if the latter only take values 0 or 1. In fuzzy set theory, classical bivalent sets are usually called crisp sets. The fuzzy set theory can be used in a wide range of domains in which information is incomplete or imprecise, such as bioinformatics. In 1965, Zadeh [24] introduced the notion of fuzzy sets. In 1971, Rosenfled [22] introduced fuzzy sets in the realm of group theory and formulated the concepts of fuzzy subgroups of a group. The concept of intuitionistic fuzzy set was introduced by Atanassov [3], as a generalization of the notion of fuzzy set. Zhan and Tan [25] defined intuitionistic fuzzy subgroup as a generalization of Rosenfeld's fuzzy subgroup. In mathematics, a t-norm (also T-norm or, unabbreviated, triangular norm) is a kind of binary operation used in the framework of probabilistic metric spaces and in multi-valued logic, specifically in fuzzy logic. A

*Corresponding author
Email address: rasulirasul@yahoo.com (Rasul Rasuli)

Received: December 1, 2022 Revised: December 16, 2022 Accepted: December 27, 2022

t-norm generalizes intersection in a lattice and conjunction in logic. The name triangular norm refers to the fact that in the framework of probabilistic metric spaces t-norms are used to generalize triangle inequality of ordinary metric spaces. T-conorms (also called S-norms) are dual to t-norms under the order-reversing operation which assigns $1-x$ to x on $[0,1]$. Some authors considered the fuzzy subgroups with respect to a t-norm and gave some results [1, 2, 23]. The author by using norms, investigated some properties of fuzzy algebraic structures [8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21]. In this paper, we define the notion of intuitionistic fuzzy subgroups on direct product of groups under norms (t-norm T and s-norm S) and investigate some properties of them. Later, we define the composion, inverse and intersection of them and obtain some results about them. Also we introduce the normality of them and give characterizations about them. Finally, under group homomorphisms, we investigate image and pre image of them and provide some results.

2. Preliminaries

This section contains some basic definitions and preliminary results which will be needed in the sequel.

Definition 2.1. (See [6]) Let H and K be groups. On the cartesian product $H \times K$, we define a binary operation by declaring $(h,k)(h_1,k_1) = (hh_1, kk_1)$ for all $(h,k),(h_1,k_1) \in H \times K$. With respect to this operation, $H \times K$ is agroup.

Definition 2.2. (See [7]) Let X be a nonempty set. A fuzzy subset of X, we mean a function from X into $[0,1]$.

Definition 2.3. (See [3]) For sets X, Y and Z, $f = (f_1, f_2) : X \to Y \times Z$ is called a complex mapping if $f_1 : X \to Y$ and $f_2 : X \to Z$ are mappings.

Definition 2.4. (See [3]) Let X be a nonempty set. A complex mapping $A = (\mu_A, \nu_A) : X \to [0,1] \times [0,1]$ is called an intuitionistic fuzzy set (in short, IFS) in X if $\mu_A + \nu_A \leqslant 1$ where the mappings $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $\nu_A(x)$) for each $x \in X$ to A, respectively. In particular $0_\sim$ and $1_\sim$ denote the intuitionistic fuzzy empty set and intuitionistic fuzzy whole set in X defined by $0_\sim(x) = (0,1)$ and $1_\sim(x) = (1,0)$, respectively. We will denote the set of all IFSs in X as $IFS(X)$.

Definition 2.5. (See [4]) Let X be a nonempty set and let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be IFSs in X. Then
(1) $A \subset B$ iff $\mu_A \leqslant \mu_B$ and $\nu_A \geqslant \nu_B$.
(2) $A = B$ iff $A \subset B$ and $B \subset A$.

Definition 2.6. (See [5]) A t-norm T is a function $T : [0,1] \times [0,1] \to [0,1]$ having the following four properties:
(T1) $T(x,1) = x$ (neutral element),
(T2) $T(x,y) \leqslant T(x,z)$ if $y \leqslant z$ (monotonicity),
(T3) $T(x,y) = T(y,x)$ (commutativity),
(T4) $T(x, T(y,z)) = T(T(x,y), z)$ (associativity),
for all $x, y, z \in [0,1]$.

It is clear that if $x_1 \geqslant x_2$ and $y_1 \geqslant y_2$, then $T(x_1, y_1) \geqslant T(x_2, y_2)$.

Example 2.7. (See [5]) (1) Standard intersection t-norm $T_m(x,y) = \min\{x,y\}$.
(2) Bounded sum t-norm $T_b(x,y) = \max\{0, x+y-1\}$.
(3) algebraic product t-norm $T_p(x,y) = xy$.
(4) Drastic t-norm

$$T_D(x,y) = \begin{cases} y & \text{if } x = 1 \\ x & \text{if } y = 1 \\ 0 & \text{otherwise.} \end{cases}$$

(5) Nilpotent minimum t-norm

$$T_{nM}(x,y) = \begin{cases} \min\{x,y\} & \text{if } x+y > 1 \\ 0 & \text{otherwise.} \end{cases}$$

(6) Hamacher product t-norm

$$T_{H_0}(x,y) = \begin{cases} 0 & \text{if } x = y = 0 \\ \frac{xy}{x+y-xy} & \text{otherwise.} \end{cases}$$

The drastic t-norm is the pointwise smallest t-norm and the minimum is the pointwise largest t-norm: $T_D(x,y) \leqslant T(x,y) \leqslant T_{\min}(x,y)$ for all $x, y \in [0,1]$.

Recall that t-norm T will be idempotent if for all $x \in [0,1]$ we have $T(x,x) = x$.

Definition 2.8. (See [5]) An s-norm S is a function $S : [0,1] \times [0,1] \to [0,1]$ having the following four properties:
(1) $S(x,0) = x$,
(2) $S(x,y) \leqslant S(x,z)$ if $y \leqslant z$,
(3) $S(x,y) = S(y,x)$,
(4) $S(x,S(y,z)) = S(S(x,y),z)$,
for all $x, y, z \in [0,1]$.

We say that S is idempotent if for all $x \in [0,1]$,$S(x,x) = x$.

Example 2.9. (See [5]) The basic s-norms are

$$S_m(x,y) = \max\{x,y\},$$

$$S_b(x,y) = \min\{1, x+y\}$$

and

$$S_p(x,y) = x + y - xy$$

for all $x, y \in [0,1]$.
S_m is standard union, S_b is bounded sum, S_p is algebraic sum.

Lemma 2.10. (See [1]) Let T be a t-norm and S be an s-norm. Then

$$T(T(x,y),T(w,z)) = T(T(x,w),T(y,z)),$$

and

$$S(S(x,y),S(w,z)) = S(S(x,w),S(y,z)),$$

for all $x, y, w, z \in [0,1]$.

Definition 2.11. (See [8]) Let $A = (\mu_A, \nu_A) \in IFS(X)$ and $B = (\mu_B, \nu_B) \in IFS(X)$. Define intesection A and B as

$$A \cap B = (\mu_A, \nu_A) \cap (\mu_B, \nu_B) = (\mu_{A\cap B}, \nu_{A\cap B})$$

such that $\mu_{A\cap B}(x) = T(\mu_A(x), \mu_B(x))$ and $\nu_{A\cap B}(x) = S(\nu_A(x), \nu_B(y))$ for all $x \in X$.

Definition 2.12. (See [8]) Let Let φ be a function from set X into set Y such that $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be two intuitionistic fuzzy sets in X and Y respectively.
For all $x \in X, y \in Y$, we define $\varphi(A) = (\varphi(\mu_A), \varphi(\nu_A)) : Y \to [0,1] \times [0,1]$
with

$$\varphi(A)(y) = (\varphi(\mu_A)(y), \varphi(\nu_A)(y)) = \begin{cases} (\sup\{\mu_A(x) \mid x \in X, \varphi(x) = y\}, \inf\{\nu_A(x) \mid x \in X, \varphi(x) = y\}) & \text{if } \varphi^{-1}(y) \neq \emptyset \\ (0,1) & \text{if } \varphi^{-1}(y) = \emptyset. \end{cases}$$

Also we define $\varphi^{-1}(B) = (\varphi^{-1}(\mu_B), \varphi^{-1}(\nu_B)) : X \to [0,1] \times [0,1]$ as

$$\varphi^{-1}(B)(x) = (\varphi^{-1}(\mu_B)(x), \varphi^{-1}(\nu_B)(x)) = (\mu_B(\varphi(x)), \nu_B(\varphi(x))).$$

3. Main results

Definition 3.1. Let G_1 and G_2 be two groups. An $A = (\mu_A, \nu_A) \in \mathsf{IFS}(G_1 \times G_2)$ is said to be an intuitionistic fuzzy subgroup on direct product of $G_1 \times G_2$ with respect to norms(t-norm T and s-norm S) if
(1) $A((x_1,x_2)(y_1,y_2)) \supseteq (T(\mu_A(x_1,x_2), \mu_A(y_1,y_2)), S(\nu_A(x_1,x_2), \nu_A(y_1,y_2)))$,
(2) $A(x_1,x_2)^{-1}) \supseteq A(x_1,x_2)$,
for all $(x_1,x_2),(y_1,y_2) \in G_1 \times G_2$. Denote by $\mathsf{IFSDPN}(G_1 \times G_2)$, the set of all intuitionistic fuzzy subgroups on direct product of $G_1 \times G_2$ with respect to norms(t-norm T and s-norm S).

Remark 3.2. Conditions (1) and (2) of Definition 3.1 are equivalent to following conditions:
(1) $\mu_A((x_1,x_2)(y_1,y_2)) \geqslant T(\mu_A(x_1,x_2), \mu_A(y_1,y_2))$,
(2) $\mu_A(x_1,x_2)^{-1} \geqslant \mu_A(x_1,x_2)$,
(3)$\nu_A((x_1,x_2)(y_1,y_2)) \leqslant S(\nu_A(x_1,x_2), \nu_A(y_1,y_2))$,
(4) $\nu_A(x_1,x_2)^{-1} \leqslant \nu_A(x_1,x_2)$,
for all $(x_1,x_2),(y_1,y_2) \in G_1 \times G_2$.

Example 3.3. Let $\mathbb{Z}_2 = \{\bar{0},\bar{1}\}, \mathbb{Z}_3 = \{\bar{0},\bar{1},\bar{2}\}$ be two additive groups. Then

$$\mathbb{Z}_2 \times \mathbb{Z}_3 = \{(\bar{0},\bar{0}),(\bar{0},\bar{1}),(\bar{0},\bar{2}),(\bar{1},\bar{0}),(\bar{1},\bar{1}),(\bar{1},\bar{2})\}.$$

Define fuzzy set $A = (\mu_A, \nu_A) : \mathbb{Z}_2 \times \mathbb{Z}_3 \to [0,1]$ by

$$\mu_A(x,y) = \begin{cases} 0.45 & \text{if } (x,y) = (\bar{0},\bar{0}) \\ 0.25 & \text{if } (x,y) = (\bar{1},\bar{0}) \\ 0.30 & \text{if } (x,y) = (\bar{0},\bar{1}) = (\bar{0},\bar{2}) \\ 0.65 & \text{if } (x,y) = (\bar{1},\bar{1}) = (\bar{1},\bar{2}) \end{cases}$$

and

$$\nu_A(x,y) = \begin{cases} 0.35 & \text{if } (x,y) = (\bar{0},\bar{0}) \\ 0.65 & \text{if } (x,y) = (\bar{1},\bar{0}) \\ 0.50 & \text{if } (x,y) = (\bar{0},\bar{1}) = (\bar{0},\bar{2}) \\ 0.15 & \text{if } (x,y) = (\bar{1},\bar{1}) = (\bar{1},\bar{2}) \end{cases}$$

for all $(x,y) \in \mathbb{Z}_2 \times \mathbb{Z}_3$. Let $T(a,b) = T_p(a,b) = ab$ and $S(a,b) = S_p(a,b) = a+b-ab$ for all $a,b \in [0,1]$. Then $A = (\mu_A, \nu_A) \in \mathsf{IFSDPN}(\mathbb{Z}_2 \times \mathbb{Z}_3)$.

Lemma 3.4. Let $A = (\mu_A, \nu_A) \in \mathsf{IFS}(G \times G)$ such that G is finite group and T and S be idempotent. If $A = (\mu_A, \nu_A)$ satisfies condition (1) of Definition 3.1, then $A = (\mu_A, \nu_A) \in \mathsf{IFSDPN}(G \times G)$.

Proof. As G is finite so we have an $(x, y) \in G \times G$ such that $(x, y) \neq (e, e)$ and (x, y) has finite order, say $n > 1$ then $(x, y)^n = (e, e)$ and $(x, y)^{-1} = (x, y)^{n-1}$. Now by using (1) repeatedly, we get that

$$\begin{aligned}\mu_A(x,y)^{-1} &= \mu_A(x,y)^{n-1} \\ &= \mu_A((x,y)^{n-2}(x,y)) \\ &\geqslant T(\mu_A((x,y)^{n-2}), \mu_A(x,y)) \\ &\geqslant T(\underbrace{\mu_A(x,y), \mu_A(x,y), ..., \mu_A((x,y)}_{n-1}) \\ &= \mu_A(x,y).\end{aligned}$$

Also

$$\begin{aligned}\nu_A(x,y)^{-1} &= \nu_A(x,y)^{n-1} \\ &= \nu_A((x,y)^{n-2}(x,y)) \leqslant (\nu_A(x,y)^{n-2}, \nu_A(x,y)) \\ &\leqslant S(\underbrace{\nu_A(x,y), \nu_A(x,y), ..., \nu_A(x,y)}_{n-1}) \\ &= \nu_A(x,y).\end{aligned}$$

Thus

$$A(x,y)^{-1} = (\mu_A(x,y)^{-1}, \nu_A(x,y)^{-1}) \supseteq (\mu_A(x,y), \nu_A(x,y)) = A(x,y)$$

and so $A = (\mu_A, \nu_A) \in IFSDPN(G \times G)$. □

Proposition 3.5. Let $A = (\mu_A, \nu_A) \in IFSDPN(G \times G)$ and T and S be idempotent.Then for all $(x, y) \in G \times G$, and $n \geqslant 1$,
(1) $A(e, e) \supseteq A(x, y)$;
(2) $A(x, y)^n \supseteq A(x, y)$;
(3) $A(x, y) = A(x, y)^{-1}$.

Proof. Let $(x, y) \in G \times G$ and $n \geqslant 1$.
(1)

$$\mu_A(e,e) = \mu_A((x,y)(x,y)^{-1}) \geqslant T(\mu_A(x,y), \mu_A(x,y)^{-1}) \geqslant T(\mu_A(x,y), \mu_A(x,y)) = \mu_A(x,y)$$

and

$$\nu_A(e,e) = \nu_A((x,y)(x,y)^{-1}) \leqslant S(\nu_A(x,y), \nu_A((x,y)^{-1})) \leqslant S(\nu_A(x,y), \nu_A(x,y)) = \nu_A(x,y)$$

and then

$$A(e,e) = (\mu_A(e,e), \nu_A(e,e)) \supseteq (\mu_A(x,y), \nu_A(x,y)) = A(x,y).$$

(2)

$$\mu_A(x,y)^n = \mu_A(\underbrace{(x,y)(x,y)...(x,y)}_{n}) \geqslant T(\underbrace{\mu_A(x,y), \mu_A(x,y), ..., \mu_A(x,y)}_{n}) = \mu_A(x,y)$$

and

$$\nu_A(x,y)^n = \nu_A(\underbrace{(x,y)(x,y)...(x,y)}_{n}) \leqslant S(\underbrace{\nu_A(x,y), \nu_A(x,y), ..., \nu_A(x,y)}_{n}) = \nu_A(x,y)$$

and thus

$$A(x,y)^n = (\mu_A(x,y)^n, \nu_A(x,y)^n) \supseteq (\mu_A(x,y), \nu_A(x,y)) = A(x,y).$$

(3) As

$$\mu_A(x,y) = \mu_A((x,y)^{-1})^{-1} \geqslant \mu_A(x,y)^{-1} \geqslant \mu_A(x,y)$$

and

$$\nu_A(x,y) = \nu_A((x,y)^{-1})^{-1} \leqslant \nu_A(x,y)^{-1} \leqslant \nu_A(x,y) = \nu_A(x,y)^{-1}$$

so $\mu_A(x,y) = \mu_A(x^{-1})$ and $\nu_A(x,y) = \nu_A(x,y)^{-1}$ and therefore

$$A(x,y) = (\mu_A(x,y), \nu_A(x,y)) = (\mu_A(x,y)^{-1}, \nu_A(x,y)^{-1}) = A(x,y)^{-1}.$$

□

Proposition 3.6. Let $A = (\mu_A, \nu_A) \in \mathsf{IFSDPN}(G \times G)$ and T and S be idempotent. Then $A((x,y)(z,t)) = A(z,t)$ if and only if $A(x,y) = A(e,e)$ for all $(x,y),(z,t) \in G \times G$.

Proof. Let $A((x,y)(z,t)) = A(z,t)$ for all $(x,y),(z,t) \in G \times G$. If we let $(z,t) = (e,e)$, then we get that $A(x,y) = A(e,e)$.
Conversely, suppose that $A(x,y) = A(e,e)$ so from Proposition 3.5 (part 1) we get that $A(x,y) \supseteq A(z,t)$ and $A(x,y) \supseteq A((x,y)(z,t))$ which mean that $\mu_A(x,y) \geqslant \mu_A(z,t)$ and $\mu_A(x,y) \geqslant \mu_A((x,y)(z,t))$ and $\nu_A(x,y) \leqslant \nu_A(z,t)$ and $\nu_A(x,y) \leqslant \nu_A((x,y)(z,t))$. Then

$$\begin{aligned}
\mu_A((x,y)(z,t)) &\geqslant T(\mu_A(x,y), \mu_A(z,t)) \\
&\geqslant T(\mu_A(z,t), \mu_A(z,t)) \\
&= \mu_A(z,t) \\
&= \mu_A((x,y)^{-1}(x,y)(z,t)) \\
&\geqslant T(\mu_A(x,y), \mu_A((x,y)(z,t))) \\
&\geqslant T(\mu_A((x,y)(z,t)), \mu_A((x,y)(z,t))) \\
&= \mu_A((x,y)(z,t))
\end{aligned}$$

and so

$$\mu_A((x,y)(z,t)) = \mu_A(z,t). \qquad \text{(a)}$$

Also

$$\begin{aligned}
\nu_A((x,y)(z,t)) &\leqslant S(\nu_A(x,y), \nu_A(z,t)) \\
&\leqslant S(\nu_A(z,t), \nu_A(z,t)) \\
&= \nu_A(z,t) \\
&= \nu_A((x,y)^{-1}(x,y)(z,t)) \\
&\leqslant S(\nu_A(x,y), \nu_A((x,y)(z,t))) \\
&\leqslant S(\nu_A((x,y)(z,t)), \nu_A((x,y)(z,t))) \\
&= \nu_A((x,y)(z,t))
\end{aligned}$$

and so

$$\nu_A((x,y)(z,t)) = \nu_A(z,t). \qquad \text{(b)}$$

Thus

$$A((x,y)(z,t)) = (\mu_A((x,y)(z,t)), \nu_A((x,y)(z,t))) = (\mu_A(z,t), \nu_A(z,t)) = A(z,t).$$

□

Definition 3.7. Let $A = (\mu_A, \nu_A) \in \mathsf{IFS}(G \times G)$ and $B = (\mu_B, \nu_B) \in \mathsf{IFS}(G \times G)$. We define the composion of A and B as $A \circ B \in \mathsf{IFS}(G \times G)$ such that for all $(x, y) \in G \times G$ we have

$$(A \circ B)(x, y) = ((\mu_A, \nu_A) \circ (\mu_B, \nu_B))(x, y) = (\mu_{A \circ B}(x, y), \nu_{A \circ B}(x, y))$$

such that

$$\mu_{A\circ B}(x,y) = \begin{cases} \sup_{(x,y)=(z,t)(m,n)} T((\mu_A(z,t), \mu_A(m,n)) & \text{if } (x,y) = (z,t)(m,n) \\ 0 & \text{if } (x,y) \neq (z,t)(m,n) \end{cases}$$

and

$$\nu_{A\circ B}(x,y) = \begin{cases} \inf_{(x,y)=(z,t)(m,n)} S((\nu_A(z,t), \nu_A(m,n)) & \text{if } (x,y) = (z,t)(m,n) \\ 0 & \text{if } (x,y) \neq (z,t)(m,n). \end{cases}$$

Proposition 3.8. Let $A^{-1} = (\mu_A^{-1}, \nu_A^{-1}) \in \mathsf{IFS}(G \times G)$ be the inverse of $A = (\mu_A, \nu_A) \in \mathsf{IFS}(G \times G)$ such that for all $(x, y) \in G \times G$ and

$$A^{-1}(x,y) = (\mu_A^{-1}(x,y), \nu_A^{-1}(x,y)) = (\mu_A(x,y)^{-1}, \nu_A(x,y)^{-1}) = A(x,y)^{-1}.$$

Let T and S be idempotent then $A \in \mathsf{IFSDPN}(G \times G)$ if and only if A satisfies the following conditions:
(1) $A \circ A \subseteq A$;
(2) $A^{-1} = A$.

Proof. Let $(x, y), (z, t), (m, n) \in G \times G$ such that $(x, y) = (z, t)(m, n)$.
If $A = (\mu_A, \nu_A) \in \mathsf{IFSDPN}(G \times G)$, then

$$\mu_A(x,y) = \mu_A((z,t)(m,n)) \geqslant T(\mu_A(z,t), \mu_A(m,n)) = \mu_{A\circ A}(x,y)$$

and

$$\nu_A(x,y) = \nu_A((z,t)(m,n)) \leqslant S(\nu_A(z,t), \nu_A(m,n)) = \nu_{A\circ A}(x,y)$$

which yield

$$(A \circ A)(x,y) = (\mu_{A\circ A}(x,y), \nu_{A\circ A}(x,y)) \subseteq (\mu_A(x,y), \nu_A(x,y)) = A(x,y)$$

and then $A \circ A \subseteq A$.
Also $A^{-1} = A$ comes from Proposition 3.5 (part3).
Conversely, let $A \circ A \subseteq A$ and $A^{-1} = A$. As $A \circ A \subseteq A$ so

$$\begin{aligned} \mu_A((z,t)(m,n)) &= \mu_A(x,y) \\ &\geqslant \mu_{A\circ A}(x,y) \\ &= \sup_{(x,y)=(z,t)(m,n)} T(\mu_A(z,t), \mu_A(m,n)) \\ &\geqslant T(\mu_A(z,t), \mu_A(m,n)) \end{aligned}$$

and

$$\begin{aligned} \nu_A((z,t)(m,n)) &= \nu_A(x,y) \\ &\leqslant \nu_{A\circ A}(x,y) \\ &= \inf_{(x,y)=(z,t)(m,n)} S(\nu_A(z,t), \nu_A(m,n)) \\ &\leqslant S(\nu_A(z,t), \nu_A(m,n)) \end{aligned}$$

which mean that

$$\begin{aligned} A((z,t)(m,n)) &= (\mu_A((z,t)(m,n)), \nu_A((z,t)(m,n))) \\ &\supseteq (T(\mu_A(z,t), \mu_A(m,n)), S(\nu_A(z,t), \nu_A(m,n))). \quad \text{(a)} \end{aligned}$$

As $A^{-1} = A$ so

$$A(x,y) = (\mu_A(x,y), \nu_A(x,y)) = (\mu_A^{-1}(x,y), \nu_A^{-1}(x,y)) = A^{-1}(x,y). \quad \text{(b)}$$

Therefore from (a) and (b) we get that $A \in IFSDPN(G \times G)$. □

Corollary 3.9. Let $A = (\mu_A, \nu_A) \in IFSDPN(G \times G)$ and $B = (\mu_B, \nu_B) \in IFSDPN(G \times G)$ and G be commutative group. Then $A \circ B \in IFSDPN(G \times G)$ if and only if $A \circ B = B \circ A$.

Proof. If $A, B, A \circ B \in IFSDPN(G \times G)$, then from Proposition 3.8 we get that $A^{-1} = A, B^{-1} = B$ and $(B \circ A)^{-1} = B \circ A$. Now $A \circ B = A^{-1} \circ B^{-1} = (B \circ A)^{-1} = B \circ A$.
Conversely, since $A \circ B = B \circ A$ we have

$$(A \circ B)^{-1} = (B \circ A)^{-1} = A^{-1} \circ B^{-1} = A \circ B.$$

Also

$$(A \circ B) \circ (A \circ B) = A \circ (B \circ A) \circ B = A \circ (A \circ B) \circ B = (A \circ A) \circ (B \circ B) \subseteq A \circ B.$$

Now Proposition 3.8 gives us that $A \circ B \in IFSDPN(G \times G)$. □

Proposition 3.10. Let $A = (\mu_A, \nu_A) \in IFSDPN(G \times G)$ and $B = (\mu_B, \nu_B) \in IFSDPN(G \times G)$. Then $A \cap B = (\mu_{A \cap B}, \nu_{A \cap B}) \in IFSDPN(G \times G)$.

Proof. Let $(x,y), (z,t) \in G \times G$. Then

$$\begin{aligned} \mu_{A\cap B}((x,y)(z,t)) &= T(\mu_A((x,y)(z,t)), \mu_B((x,y)(z,t))) \\ &\geqslant T(T(\mu_A(x,y), \mu_A(z,t)), T(\mu_B(x,y), \mu_B(z,t))) \\ &= T(T(\mu_A(x,y), \mu_B(x,y)), T(\mu_A(z,t), \mu_B(z,t))) \quad \text{(Lemma 2.10)} \\ &= T(\mu_{A\cap B}(x,y), \mu_{A\cap B}(z,t)) \end{aligned}$$

and

$$\begin{aligned} \nu_{A\cap B}((x,y)(z,t)) &= S(\nu_A((x,y)(z,t)), \nu_B((x,y)(z,t))) \\ &\leqslant S(S(\nu_A(x,y), \nu_A(z,t)), S(\nu_B(x,y), \nu_B(z,t))) \\ &= S(S(\nu_A(x,y), \nu_B(x,y)), S(\nu_A(z,t), \nu_B(z,t))) \quad \text{(Lemma 2.10)} \\ &= S(\nu_{A\cap B}(x,y), \nu_{A\cap B}(z,t)) \end{aligned}$$

which mean that

$$\begin{aligned} (A\cap B)((x,y)(z,t)) &= (\mu_{A\cap B}((x,y)(z,t)), \nu_{A\cap B}((x,y)(z,t))) \\ &\supseteq (T(\mu_{A\cap B}(x,y), \mu_{A\cap B}(z,t)), S(\nu_{A\cap B}(x,y), \nu_{A\cap B}(z,t))). \end{aligned}$$

Also

$$\mu_{A\cap B}((x,y)^{-1}) = T(\mu_A((x,y)^{-1}), \mu_B((x,y)^{-1})) \geqslant T(\mu_A(x,y), \mu_B(x,y)) = \mu_{A\cap B}(x,y)$$

and

$$\nu_{A\cap B}((x,y)^{-1}) = S(\nu_A((x,y)^{-1}), \nu_B((x,y)^{-1})) \leqslant S(\nu_A(x,y), \nu_B(x,y)) = \nu_{A\cap B}(x,y)$$

so

$$(A\cap B)((x,y)^{-1}) = (\mu_{A\cap B}((x,y)^{-1}), \nu_{A\cap B}((x,y)^{-1})) \supseteq (\mu_{A\cap B}(x,y), \nu_{A\cap B}(x,y)) = (A\cap B)(x,y).$$

Thus $A\cap B = (\mu_{A\cap B}, \nu_{A\cap B}) \in \mathsf{IFSDPN}(G\times G)$. □

Corollary 3.11. Let $I_n = \{1,2,...,n\}$. If $\{A_i = (\mu_{A_i}, \nu_{A_i}) \mid i \in I_n\} \subseteq \mathsf{IFSDPN}(G\times G)$. Then

$$A = \cap_{i\in I_n} A_i \in \mathsf{IFSDPN}(G\times G).$$

Definition 3.12. We say that $A = (\mu_A, \nu_A) \in \mathsf{IFSDPN}(G\times G)$ is normal if for all $(x,y),(z,t) \in G\times G$ we have that $A((x,y)(z,t)(x,y)^{-1}) = A(z,t)$. Also we denote by $\mathsf{NIFSDPN}(G\times G)$ the set of all normal intuitionistic fuzzy subgroups on direct product of $G_1 \times G_2$ with respect to norms(t-norm T and s-norm S).

Proposition 3.13. Let $A = (\mu_A, \nu_A) \in \mathsf{NIFSDPN}(G\times G)$ and $B = (\mu_B, \nu_B) \in \mathsf{NIFSDPN}(G\times G)$. Then $A\cap B = (\mu_{A\cap B}, \nu_{A\cap B}) \in \mathsf{NIFSDPN}(G\times G)$.

Proof. As Proposition 3.10 we have that $A\cap B = (\mu_{A\cap B}, \nu_{A\cap B}) \in \mathsf{IFSDPN}(G\times G)$. Let $(x,y),(z,t) \in G\times G$ then

$$\begin{aligned}\mu_{A\cap B}((x,y)(z,t)(x,y)^{-1}) &= T(\mu_A((x,y)(z,t)(x,y)^{-1}), \mu_B((x,y)(z,t)(x,y)^{-1}))\\ &= T(\mu_A(z,t), \mu_B(z,t))\\ &= \mu_{A\cap B}(z,t)\end{aligned}$$

and

$$\begin{aligned}\nu_{A\cap B}((x,y)(z,t)(x,y)^{-1}) &= S(\nu_A((x,y)(z,t)(x,y)^{-1}), \nu_B((x,y)(z,t)(x,y)^{-1}))\\ &= S(\nu_A(z,t), \nu_B(z,t))\\ &= \nu_{A\cap B}(z,t)\end{aligned}$$

and thus

$$\begin{aligned}(A\cap B)((x,y)(z,t)(x,y)^{-1}) &= (\mu_{A\cap B}((x,y)(z,t)(x,y)^{-1}), \nu_{A\cap B}((x,y)(z,t)(x,y)^{-1}))\\ &= (\mu_{A\cap B}(z,t), \nu_{A\cap B}(z,t))\\ &= (A\cap B)(z,t).\end{aligned}$$

Therefore $A\cap B = (\mu_{A\cap B}, \nu_{A\cap B}) \in \mathsf{NIFSDPN}(G\times G)$. □

Corollary 3.14. Let $I_n = \{1,2,...,n\}$. If $\{A_i = (\mu_{A_i}, \nu_{A_i}) \mid i \in I_n\} \subseteq \mathsf{NIFSDPN}(G\times G)$. Then

$$A = \cap_{i\in I_n} \mu_i \in \mathsf{NIFSDPN}(G\times G).$$

Definition 3.15. Let $A = (\mu_A, \nu_A) \in \mathsf{IFSDPN}(G\times G)$ and $B = (\mu_B, \nu_B) \in \mathsf{IFSDPN}(G\times G)$ such that $A \subseteq B$. Then A is called normal in B, written $A \blacktriangleright B$, if for all $(x,y),(z,t) \in G\times G$ we have

$$\begin{aligned}A((x,y)(z,t)(x,y)^{-1}) &= (\mu_A((x,y)(z,t)(x,y)^{-1}), \nu_A((x,y)(z,t)(x,y)^{-1}))\\ &\supseteq (T(\mu_A(z,t), \mu_B(x,y)), S(\nu_A(z,t), \nu_B(x,y))).\end{aligned}$$

Proposition 3.16. If T and S be idempotent, then every intuitionistic fuzzy subgroup on direct product of $G \times G$ with respect to norms(t-norm T and s-norm S) is normal fuzzy subgroup in itself.

Proof. Let $A = (\mu_A, \nu_A) \in IFSDPN(G \times G)$ and $(x, y), (z, t) \in G \times G$ then

$$\begin{aligned}
\mu_A((x,y)(z,t)(x,y)^{-1}) &\geqslant T(\mu_A((x,y)(z,t)), \mu_A((x,y)^{-1})) \\
&\geqslant T(T(\mu_A(x,y), \mu_A(z,t)), \mu_A(x,y)) \\
&= T(T(\mu_A(z,t), \mu_A(x,y)), \mu_A(x,y)) \\
&= T(\mu_A(z,t), T(\mu_A(x,y), \mu_A(x,y))) \\
&= T(\mu_A(z,t), \mu_A(x,y))
\end{aligned}$$

and

$$\begin{aligned}
\nu_A((x,y)(z,t)(x,y)^{-1}) &\leqslant S(\nu_A(xy), \nu_A(x^{-1})) \\
&\leqslant S(S(\nu_A(x), \nu_A(y)), \nu_A(x)) \\
&= S(S(\nu_A(z,t), \nu_A(x,y)), \nu_A(x,y)) \\
&= S(\nu_A(z,t), S(\nu_A(x,y), \nu_A(x,y))) \\
&= S(\nu_A(z,t), \nu_A(x,y)).
\end{aligned}$$

Thus

$$\begin{aligned}
A((x,y)(z,t)(x,y)^{-1}) &= (\mu_A((x,y)(z,t)(x,y)^{-1}), \nu_A((x,y)(z,t)(x,y)^{-1})) \\
&\supseteq (T(\mu_A(z,t), \mu_A(x,y)), S(\nu_A(z,t), \nu_B(x,y)))
\end{aligned}$$

so $A = (\mu_A, \nu_A) \blacktriangleright A = (\mu_A, \nu_A)$. □

Proposition 3.17. Let $A = (\mu_A, \nu_A) \in NIFSDPN(G \times G)$ and $B = (\mu_B, \nu_B) \in IFSDPN(G \times G)$ such that T and S be idempotent. Then $A \cap B \blacktriangleright B$.

Proof. By Proposition 3.10 we get that $A \cap B \in IFSDPN(G \times G)$. Let $(x, y), (z, t) \in G \times G$ then

$$\begin{aligned}
\mu_{A\cap B}((x,y)(z,t)(x,y)^{-1}) &= T(\mu_A((x,y)(z,t)(x,y)^{-1}), \mu_B((x,y)(z,t)(x,y)^{-1})) \\
&= T(\mu_A(z,t), \mu_B((x,y)(z,t)(x,y)^{-1})) \\
&\geqslant T(\mu_A(z,t), T(\mu_B((x,y)(z,t)), \mu_B((x,y)^{-1})) \\
&\geqslant T(\mu_A(z,t), T(T(\mu_B(x,y), \mu_B(z,t)), \mu_B(x,y))) \\
&= T(\mu_A(z,t), T(\mu_B(z,t), T(\mu_B(x,y), \mu_B(x,y)))) \\
&= T(\mu_A(z,t), T(\mu_B(z,t), \mu_B(x,y))) \\
&= T(T(\mu_A(z,t), \mu_B(z,t)), \mu_B(x,y)) \\
&= T(\mu_{A\cap B}(z,t), \mu_B(x,y))
\end{aligned}$$

and

$$\begin{aligned}\nu_{A\cap B}((x,y)(z,t)(x,y)^{-1}) &= S(\nu_A((x,y)(z,t)(x,y)^{-1}), \nu_B((x,y)(z,t)(x,y)^{-1}))\\ &= S(\nu_A(z,t), \nu_B((x,y)(z,t)(x,y)^{-1}))\\ &\leqslant S(\nu_A(z,t), S(\nu_B((x,y)(z,t)), \nu_B((x,y)^{-1}))\\ &\leqslant S(\nu_A(z,t), S(S(\nu_B(x,y), \nu_B(z,t)), \nu_B(x,y)))\\ &= S(\nu_A(z,t), S(\nu_B(z,t), S(\nu_B(x,y), \nu_B(x,y))))\\ &= S(\nu_A(z,t), S(\nu_B(z,t), \nu_B(x,y)))\\ &= S(S(\nu_A(z,t), \nu_B(z,t)), \nu_B(x,y))\\ &= S(\nu_{A\cap B}(z,t), \nu_B(x,y))\end{aligned}$$

and thus

$$\begin{aligned}(A\cap B)(xyx^{-1}) &= (\mu_{A\cap B}(xyx^{-1}), \nu_{A\cap B}(xyx^{-1}))\\ &\supseteq (T(\mu_{A\cap B}(z,t), \mu_B(x,y)), S(\nu_{A\cap B}(z,t), \nu_B(x,y)))\end{aligned}$$

which means that $A \cap B \blacktriangleright B$. □

Proposition 3.18. Let $A = (\mu_A, \nu_A) \in IFSDPN(G\times G)$ and $B = (\mu_B, \nu_B) \in IFSDPN(G\times G)$ and $C = (\mu_C, \nu_C) \in IFSDPN(G\times G)$ such that T and S be idempotent. If $A \blacktriangleright C$ and $B \blacktriangleright C$, then $A\cap B \blacktriangleright C$.

Proof. As Proposition 3.10 we will have that $A \cap B \in IFSDPN(G\times G)$. Let $(x,y),(z,t) \in G\times G$ then

$$\begin{aligned}\mu_{A\cap B}((x,y)(z,t)(x,y)^{-1}) &= T(\mu_A((x,y)(z,t)(x,y)^{-1}), \mu_B((x,y)(z,t)(x,y)^{-1}))\\ &\geqslant T(T(\mu_A(z,t), \mu_C(x,y)), T(\mu_B(z,t), \mu_C(x,y)))\\ &= T(T(\mu_A(z,t), \mu_B(z,t)), T(\mu_C(x,y), \mu_C(x,y)))\\ &= T(T(\mu_A(z,t), \mu_B(z,t)), \mu_C(x,y))\\ &= T(\mu_{A\cap B}(z,t), \mu_C(x,y))\end{aligned}$$

and

$$\begin{aligned}\nu_{A\cap B}((x,y)(z,t)(x,y)^{-1}) &= S(\nu_A((x,y)(z,t)(x,y)^{-1}), \nu_B((x,y)(z,t)(x,y)^{-1}))\\ &\leqslant S(S(\nu_A(z,t), \nu_C(x,y)), S(\nu_B(z,t), \nu_C(x,y)))\\ &= S(S(\nu_A(z,t), \nu_B(z,t)), S(\nu_C(x,y), \nu_C(x,y)))\\ &= S(S(\nu_A(z,t), \nu_B(z,t)), \nu_C(x,y))\\ &= S(\nu_{A\cap B}(z,t), \nu_C(x,y))\end{aligned}$$

and therefore

$$\begin{aligned}(A\cap B)((x,y)(z,t)(x,y)^{-1}) &= (\mu_{A\cap B}((x,y)(z,t)(x,y)^{-1}), \nu_{A\cap B}((x,y)(z,t)(x,y)^{-1}))\\ &\supseteq (T(\mu_{A\cap B}(z,t), \mu_C(x,y)), S(\nu_{A\cap B}(z,t), \nu_C(x,y)))\end{aligned}$$

and then $A\cap B \blacktriangleright C$. □

Corollary 3.19. Let $I_n = \{1, 2, ..., n\}$ and $\{A_i = (\mu_{A_i}, \nu_{A_i}) \mid i \in I_n\} \subseteq IFSDPN(G\times G)$ such that $\{A_i = (\mu_{A_i}, \nu_{A_i}) \mid i \in I_n\} \blacktriangleright B = (\mu_B, \nu_B)$. Then

$$A = \cap_{i\in I_n} A_i \blacktriangleright B = (\mu_B, \nu_B).$$

Proposition 3.20. Let $A = (\mu_A, \nu_A) \in IFSDPN(G \times G)$ and $H \times H$ be a group. Suppose that $\varphi : G \times G \to H \times H$ be a homomorphism. Then $\varphi(A) \in IFSDPN(H \times H)$.

Proof. Let $(u_1, u_2), (v_1, v_2) \in H \times H$ and $(x_1, x_2), (y_1, y_2) \in G \times G$ such that $(u_1, u_2) = \varphi(x_1, x_2)$ and $(v_1, v_2) = \varphi(y_1, y_2)$ and $\varphi(A) = (\varphi(\mu_A), \varphi(\nu_A))$. Now

$$\begin{aligned}\varphi(\mu_A)((u_1,u_2)(v_1,v_2)) &= \sup\{\mu_A((x_1,x_2)(y_1,y_2)) \mid (u_1,u_2)=\varphi(x_1,x_2),(v_1,v_2)=\varphi(y_1,y_2)\}\\ &\geqslant \sup\{T(\mu_A(x_1,x_2),\mu_A(y_1,y_2)) \mid (u_1,u_2)=\varphi(x_1,x_2),(v_1,v_2)=\varphi(y_1,y_2)\}\\ &= T(\sup\{\mu_A(x_1,x_2)|(u_1,u_2)=\varphi(x_1,x_2)\},\sup\{\mu_A(y_1,y_2)|(v_1,v_2)=\varphi(y_1,y_2)\})\\ &= T(\varphi(\mu_A)(u_1,u_2),\varphi(\mu_A)(v_1,v_2))\end{aligned}$$

and thus

$$\varphi(\mu_A)((u_1,u_2)(v_1,v_2)) \geqslant T(\varphi(\mu_A)(u_1,u_2),\varphi(\mu_A)(v_1,v_2)). \quad (1)$$

Also

$$\begin{aligned}\varphi(\nu_A)((u_1,u_2)(v_1,v_2)) &= \inf\{\nu_A((x_1,x_2)(y_1,y_2)) \mid (u_1,u_2)=\varphi(x_1,x_2),(v_1,v_2)=\varphi(y_1,y_2)\}\\ &\leqslant \inf\{S(\nu_A(x_1,x_2),\nu_A(y_1,y_2)) \mid (u_1,u_2)=\varphi(x_1,x_2),(v_1,v_2)=\varphi(y_1,y_2)\}\\ &= S(\inf\{\nu_A(x_1,x_2)|(u_1,u_2)=\varphi(x_1,x_2)\},\inf\{\nu_A(y_1,y_2)|(v_1,v_2)=\varphi(y_1,y_2)\})\\ &= S(\varphi(\nu_A)(u_1,u_2),\varphi(\nu_A)(v_1,v_2))\end{aligned}$$

and so

$$\varphi(\nu_A)((u_1,u_2)(v_1,v_2)) \leqslant S(\varphi(\nu_A)(u_1,u_2),\varphi(\nu_A)(v_1,v_2)). \quad (2)$$

Since

$$\begin{aligned}\varphi(\mu_A)((u_1,u_2)^{-1}) &= \sup\{\mu_A((x_1,x_2)^{-1}) \mid (u_1,u_2)^{-1}=\varphi((x_1,x_2)^{-1})\}\\ &= \sup\{\mu_A((x_1,x_2)^{-1}) \mid (u_1,u_2)^{-1}=\varphi^{-1}(x_1,x_2)\}\\ &\geqslant \sup\{\mu_A(x_1,x_2) \mid (u_1,u_2)=\varphi(x_1,x_2)\}\\ &= \varphi(\mu_A)(u_1,u_2)\end{aligned}$$

and then

$$\varphi(\mu_A)((u_1,u_2)^{-1}) \geqslant \varphi(\mu_A)(u_1,u_2). \quad (3)$$

Finally

$$\begin{aligned}\varphi(\nu_A)((u_1,u_2)^{-1}) &= \inf\{\nu_A((x_1,x_2)^{-1}) \mid (u_1,u_2)^{-1}=\varphi((x_1,x_2)^{-1})\}\\ &= \inf\{\nu_A((x_1,x_2)^{-1}) \mid (u_1,u_2)^{-1}=\varphi^{-1}(x_1,x_2)\}\\ &\leqslant \inf\{\nu_A(x_1,x_2) \mid (u_1,u_2)=\varphi(x_1,x_2)\}\\ &= \varphi(\nu_A)(u_1,u_2)\end{aligned}$$

and thus

$$\varphi(\nu_A)((u_1,u_2)^{-1}) \leqslant \varphi(\nu_A)(u_1,u_2). \quad (4)$$

Therefore (1)-(4) give us that $\varphi(A) \in IFSDPN(H \times H)$. □

Proposition 3.21. Let $H \times H$ be a group and $B = (\mu_B, \nu_B) \in IFSDPN(H \times H)$. Suppose that $\varphi : G \times G \to H \times H$ is a homomorphism. Then $\varphi^{-1}(B) \in IFSDPN(G \times G)$.

Proof. Let $(x_1, x_2), (y_1, y_2) \in G \times G$ and $\varphi^{-1}(B) = (\varphi^{-1}(\mu_B), \varphi^{-1}(\nu_B)) = (\mu_B(\varphi), \nu_B(\varphi)$. Now

$$\begin{aligned}\varphi^{-1}(\mu_B)((x_1, x_2)(y_1, y_2)) &= \mu_B(\varphi((x_1, x_2)(y_1, y_2))) \\ &= \mu_B(\varphi(x_1, x_2)\varphi(y_1, y_2)) \\ &\geqslant T(\mu_B(\varphi(x_1, x_2)), \mu_B(\varphi(y_1, y_2))) \\ &= T(\varphi^{-1}(\mu_B)(x_1, x_2), \varphi^{-1}(\mu_B)(y_1, y_2))\end{aligned}$$

thus

$$\varphi^{-1}(\mu_B)((x_1, x_2)(y_1, y_2)) \geqslant T(\varphi^{-1}(\mu_B)(x_1, x_2), \varphi^{-1}(\mu_B)(y_1, y_2)). \quad (1)$$

As

$$\begin{aligned}\varphi^{-1}(\nu_B)((x_1, x_2)(y_1, y_2)) &= \nu_B(\varphi((x_1, x_2)(y_1, y_2))) \\ &= \nu_B(\varphi(x_1, x_2)\varphi(y_1, y_2)) \\ &\leqslant S(\nu_B(\varphi(x_1, x_2)), \nu_B(\varphi(y_1, y_2))) \\ &= S(\varphi^{-1}(\nu_B)(x_1, x_2), \varphi^{-1}(\nu_B)(y_1, y_2))\end{aligned}$$

thus

$$\varphi^{-1}(\nu_B)((x_1, x_2)(y_1, y_2)) \leqslant S(\varphi^{-1}(\nu_B)(x_1, x_2), \varphi^{-1}(\nu_B)(y_1, y_2)). \quad (2)$$

Also

$$\varphi^{-1}(\mu_B)(x_1, x_2)^{-1} = \mu_B(\varphi(x_1, x_2)^{-1}) = \mu_B(\varphi^{-1}(x_1, x_2)) \geqslant \mu_B(\varphi(x_1, x_2)) = \varphi^{-1}(\mu_B)(x_1, x_2)$$

and then

$$\varphi^{-1}(\mu_B)(x_1, x_2)^{-1} \geqslant \varphi^{-1}(\mu_B)(x_1, x_2). \quad (3)$$

As

$$\varphi^{-1}(\nu_B)(x_1, x_2)^{-1} = \nu_B(\varphi(x_1, x_2)^{-1}) = \nu_B(\varphi^{-1}(x_1, x_2)) \leqslant \nu_B(\varphi(x_1, x_2)) = \varphi^{-1}(\nu_B)(x_1, x_2)$$

and then

$$\varphi^{-1}(\nu_B)(x_1, x_2)^{-1} \leqslant \varphi^{-1}(\nu_B)(x_1, x_2). \quad (4)$$

Therefore from (1)-(4) we get that $\varphi^{-1}(B) \in \mathsf{IFSDPN}(G \times G)$. □

Proposition 3.22. Let $A = (\mu_A, \nu_A) \in \mathsf{NIFSDPN}(G \times G)$ and H be a group. Suppose that $\varphi : G \times G \to H \times H$ be a homomorphism. Then $\varphi(A) \in \mathsf{NIFSDPN}(H \times H)$.

Proof. By Proposition 3.20 we get that $\varphi(A) \in \mathsf{IFSDPN}(H \times H)$. Let $(x_1, x_2), (y_1, y_2) \in H \times H$ such that $\varphi(u_1, u_2) = (x_1, x_2)$ and $\varphi(w_1, w_2) = (y_1, y_2)$ with $(u_1, u_2), (w_1, w_2) \in G \times G$. Then
$\varphi(\mu_A((x_1, x_2)(y_1, y_2)(x_1, x_2)^{-1})) = \sup\{\mu_A(w_1, w_2) \mid (w_1, w_2) \in G \times G, \varphi(w_1, w_2) = (x_1, x_2)(y_1, y_2)(x_1, x_2)^{-1}\}$
$= \sup\{\mu_A(w_1, w_2) \mid (w_1, w_2) \in G \times G, \varphi(w_1, w_2) = \varphi(u_1, u_2)\varphi(w_1, w_2)\varphi((u_1, u_2)^{-1})\}$
$= \sup\{\mu_A(w_1, w_2) \mid (w_1, w_2) \in G \times G, \varphi(w_1, w_2) = \varphi((u_1, u_2)(w_1, w_2)(u_1, u_2)^{-1})\}$
$= \sup\{\mu_A((u_1, u_2)(w_1, w_2)(u_1, u_2)^{-1}) \mid (w_1, w_2) \in G \times G, \varphi((u_1, u_2)(w_1, w_2)(u_1, u_2)^{-1}) = (y_1, y_2)\}$
$= \sup\{\mu_A(w_1, w_2) \mid (w_1, w_2) \in G \times G, \varphi(w_1, w_2) = (y_1, y_2)\} = \varphi(\mu_A(y_1, y_2))$
and
$\varphi(\nu_A((x_1, x_2)(y_1, y_2)(x_1, x_2)^{-1})) = \inf\{\nu_A(w_1, w_2) \mid (w_1, w_2) \in G \times G, \varphi(w_1, w_2) = (x_1, x_2)(y_1, y_2)(x_1, x_2)^{-1}\}$
$= \inf\{\nu_A(w_1, w_2) \mid (w_1, w_2) \in G \times G, \varphi(w_1, w_2) = \varphi(u_1, u_2)\varphi(w_1, w_2)\varphi((u_1, u_2)^{-1})\} = \inf\{\nu_A(w_1, w_2) \mid (w_1, w_2) \in G \times G, \varphi(w_1, w_2) = \varphi((u_1, u_2)(w_1, w_2)(u_1, u_2)^{-1})\}$
$= \inf\{\nu_A((u_1, u_2)(w_1, w_2)(u_1, u_2)^{-1}) \mid (w_1, w_2) \in G \times G, \varphi((u_1, u_2)(w_1, w_2)(u_1, u_2)^{-1}) = (y_1, y_2)\}$
$= \inf\{\nu_A(w_1, w_2) \mid (w_1, w_2) \in G \times G, \varphi(w_1, w_2) = (y_1, y_2)\} = \varphi(\nu_A(y_1, y_2))$
so

$$\begin{aligned}\varphi(A)(x_1,x_2)(y_1,y_2)(x_1,x_2)^{-1} &= (\varphi(\mu_A(x_1,x_2)(y_1,y_2)(x_1,x_2)^{-1})), \varphi(\nu_A(x_1,x_2)(y_1,y_2)(x_1,x_2)^{-1}))) \\ &= (\varphi(\mu_A(y_1,y_2)), \varphi(\nu_A(y_1,y_2))) \\ &= \varphi(A)(y_1,y_2))\end{aligned}$$

Then $\varphi(A) \in \mathrm{NIFSDPN}(H \times H)$. □

Proposition 3.23. Let $H \times H$ be a group and $B = (\mu_B, \nu_B) \in \mathrm{NIFSDPN}(H \times H)$. Suppose that $\varphi : G \times G \to H \times H$ be a homomorphism. Then $\varphi^{-1}(B) \in \mathrm{NIFSDPN}(G \times G)$.

Proof. From Proposition 3.21 we get that $\varphi^{-1}(B) \in \mathrm{IFSDPN}(G \times G)$. Let $(x_1, x_2), (y_1, y_2) \in G \times G$ then

$$\begin{aligned}\varphi^{-1}(\mu_B)((x_1,x_2)(y_1,y_2)(x_1,x_2)^{-1}) &= \mu_B(\varphi((x_1,x_2)(y_1,y_2)(x_1,x_2)^{-1})) \\ &= \mu_B(\varphi(x_1,x_2)\varphi(y_1,y_2)\varphi((x_1,x_2)^{-1})) \\ &= \mu_B(\varphi(x_1,x_2)\varphi(y_1,y_2)\varphi^{-1}(x_1,x_2)) \\ &= \mu_B(\varphi(y_1,y_2)) \\ &= \varphi^{-1}(\mu_B)(y_1,y_2)\end{aligned}$$

and

$$\begin{aligned}\varphi^{-1}(\nu_B)((x_1,x_2)(y_1,y_2)(x_1,x_2)^{-1}) &= \nu_B(\varphi((x_1,x_2)(y_1,y_2)(x_1,x_2)^{-1})) \\ &= \nu_B(\varphi(x_1,x_2)\varphi(y_1,y_2)\varphi((x_1,x_2)^{-1})) \\ &= \nu_B(\varphi(x_1,x_2)\varphi(y_1,y_2)\varphi^{-1}(x_1,x_2)) \\ &= \nu_B(\varphi(y_1,y_2)) \\ &= \varphi^{-1}(\nu_B)(y_1,y_2)\end{aligned}$$

Thus

$$\begin{aligned}\varphi^{-1}(B)(x_1,x_2)(y_1,y_2)(x_1,x_2)^{-1}) &= (\varphi^{-1}(\mu_B)(x_1,x_2)(y_1,y_2)(x_1,x_2)^{-1}), \varphi^{-1}(\nu_B)(x_1,x_2)(y_1,y_2)(x_1,x_2)^{-1})) \\ &= (\varphi^{-1}(\mu_B)(y_1,y_2), \varphi^{-1}(\nu_B)(y_1,y_2)) \\ &= \varphi^{-1}(B)(y_1,y_2).\end{aligned}$$

Then $\varphi^{-1}(B) \in \mathrm{NIFSDPN}(G \times G)$. □

Proposition 3.24. Let $A = (\mu_A, \nu_A) \in \mathrm{IFSDPN}(G \times G)$ and $B = (\mu_B, \nu_B) \in \mathrm{IFSDPN}(G \times G)$ such that $A \blacktriangleright B$. If $\varphi : G \times G \to H \times H$ be a homomorphism, then $\varphi(A) \blacktriangleright \varphi(B)$.

Proof. As Proposition 3.20 we obtain that $\varphi(A) \in \mathrm{IFSDPN}(H \times H)$ and $\varphi(B) \in \mathrm{IFSDPN}(H \times H)$. Let $(x_1, x_2), (y_1, y_2) \in H \times H$ and $(u_1, u_2), (v_1, v_2) \in G \times G$ then
$\varphi(\mu_A)((x_1,x_2)(y_1,y_2)(x_1,x_2)^{-1}) = \sup\{\mu_A(z_1,z_2) \mid (z_1,z_2) \in G \times G, \varphi(z_1,z_2) = (x_1,x_2)(y_1,y_2)(x_1,x_2)^{-1}\}$
$= \sup\{\mu_A((u_1,u_2)(v_1,v_2)(u_1,u_2)^{-1}) \mid (u_1,u_2),(v_1,v_2) \in G \times G, \varphi(u_1,u_2) = (x_1,x_2), \varphi(v_1,v_2) = (y_1,y_2)\}$
$\geqslant \sup\{T(\mu_A(v_1,v_2), \mu_B(u_1,u_2)) \mid \varphi(u_1,u_2) = (x_1,x_2), \varphi(v_1,v_2) = y\}$
$= T(\sup\{\mu_A(v_1,v_2) \mid (y_1,y_2) = \varphi(v_1,v_2)\}, \sup\{\mu_B(u_1,u_2) \mid (x_1,x_2) = \varphi(u_1,u_2)\})$
$= T(\varphi(\mu_A)(y_1,y_2), \varphi(\mu_B)(x_1,x_2))$.
Also
$\varphi(\nu_A)((x_1,x_2)(y_1,y_2)(x_1,x_2)^{-1}) = \inf\{\nu_A(z_1,z_2) \mid (z_1,z_2) \in G \times G, \varphi(z_1,z_2) = (x_1,x_2)(y_1,y_2)(x_1,x_2)^{-1}\}$
$= \inf\{\nu_A((u_1,u_2)(v_1,v_2)(u_1,u_2)^{-1}) \mid (u_1,u_2),(v_1,v_2) \in G \times G, \varphi(u_1,u_2) = (x_1,x_2), \varphi(v_1,v_2) = (y_1,y_2)\}$

$\leqslant \inf\{S(\nu_A(v_1,v_2),\nu_B(u_1,u_2)) \mid \varphi(u_1,u_2)=(x_1,x_2),\varphi(v_1,v_2)=y\}$
$= S(\inf\{\nu_A(v_1,v_2) \mid (y_1,y_2)=\varphi(v_1,v_2)\},\inf\{\nu_B(u_1,u_2) \mid (x_1,x_2)=\varphi(u_1,u_2)\})$
$= S(\varphi(\nu_A)(y_1,y_2),\varphi(\nu_B)(x_1,x_2))$.
Then

$$\begin{aligned}\varphi(A)(x_1,x_2)(y_1,y_2)(x_1,x_2)^{-1}) &= (\varphi(\mu_A)(x_1,x_2)(y_1,y_2)(x_1,x_2)^{-1}),\varphi(\nu_A)(x_1,x_2)(y_1,y_2)(x_1,x_2)^{-1}))\\ &\supseteq (T(\varphi(\mu_A)(y_1,y_2),\varphi(\mu_B)(x_1,x_2)),S(\varphi(\nu_A)(y_1,y_2),\varphi(\nu_B)(x_1,x_2))).\end{aligned}$$

Therefore $\varphi(A) \blacktriangleright \varphi(B)$. □

Proposition 3.25. Let $A=(\mu_A,\nu_A) \in \mathrm{IFSDPN}(H\times H)$ and $B=(\mu_B,\nu_B) \in \mathrm{IFSDPN}(H\times H)$ such that $A \blacktriangleright B$. If $\varphi: G\times G \to H\times H$ be a homomorphism, then $\varphi^{-1}(A) \blacktriangleright \varphi^{-1}(B)$.

Proof. Proposition 3.21 gives that $\varphi^{-1}(A) \in \mathrm{IFSDPN}(G\times G)$ and $\varphi^{-1}(B) \in \mathrm{IFSDPN}(G\times G)$. Let $(x_1,x_2),(y_1,y_2) \in G\times G$ then

$$\begin{aligned}\varphi^{-1}(\mu_A)((x_1,x_2)(y_1,y_2)(x_1,x_2)^{-1}) &= \mu_A(\varphi((x_1,x_2)(y_1,y_2)(x_1,x_2)^{-1}))\\ &= \mu_A(\varphi(x_1,x_2)\varphi(y_1,y_2)\varphi((x_1,x_2)^{-1}))\\ &= \mu_A(\varphi(x_1,x_2)\varphi(y_1,y_2)\varphi^{-1}(x_1,x_2))\\ &\geqslant T(\mu_A(\varphi(y_1,y_2)),\mu_B(\varphi(x_1,x_2)))\\ &= T(\varphi^{-1}(\mu_A)(y_1,y_2),\varphi^{-1}(\mu_B)(x_1,x_2))\end{aligned}$$

and

$$\begin{aligned}\varphi^{-1}(\nu_A)((x_1,x_2)(y_1,y_2)(x_1,x_2)^{-1}) &= \nu_A(\varphi((x_1,x_2)(y_1,y_2)(x_1,x_2)^{-1}))\\ &= \nu_A(\varphi(x_1,x_2)\varphi(y_1,y_2)\varphi((x_1,x_2)^{-1}))\\ &= \nu_A(\varphi(x_1,x_2)\varphi(y_1,y_2)\varphi^{-1}(x_1,x_2))\\ &\leqslant S(\nu_A(\varphi(y_1,y_2)),\nu_B(\varphi(x_1,x_2)))\\ &= S(\varphi^{-1}(\nu_A)(y_1,y_2),\varphi^{-1}(\nu_B)(x_1,x_2))\end{aligned}$$

thus

$$\begin{gathered}\varphi^{-1}(A)((x_1,x_2)(y_1,y_2)(x_1,x_2)^{-1})\\ = (\varphi^{-1}(\mu_A)((x_1,x_2)(y_1,y_2)(x_1,x_2)^{-1}),\varphi^{-1}(\nu_A)((x_1,x_2)(y_1,y_2)(x_1,x_2)^{-1}))\\ \supseteq (T(\varphi^{-1}(\mu_A)(y_1,y_2),\varphi^{-1}(\mu_B)(x_1,x_2)),S(\varphi^{-1}(\nu_A)(y_1,y_2),\varphi^{-1}(\nu_B)(x_1,x_2)))\end{gathered}$$

and then $\varphi^{-1}(A) \blacktriangleright \varphi^{-1}(B)$. □

Acknowledgment

We would like to thank the reviewers for carefully reading the manuscript and making several helpful comments to increase the quality of the paper.

References

[1] M. T. Abu Osman, On some products of fuzzy subgroups, Fuzzy Sets and Systems, 24 (1987), 79-86. 1, 2.10
[2] J. M. Anthony and H. Sherwood, A characterization of fuzzy subgroups, Fuzzy Sets and Systems, 7(1982), 297-305. 1
[3] K. T. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and Systems, 20(1986), 87-96. 1, 2.3, 2.4
[4] K. T. Atanassov, New operations defined over the intuitionistic fuzzy sets, Fuzzy Sets and Systems, 61(1994), 137-142. 2.5
[5] J. J. Buckley and E. Eslami, An introduction to fuzzy logic and fuzzy sets, Springer-Verlag Berlin Heidelberg GmbH, (2002). 2.6, 2.7, 2.8, 2.9
[6] T. Hungerford, Algebra, Graduate Texts in Mathematics. Springer (2003). 2.1
[7] D. S. Malik and J. N. Mordeson, Fuzzy Commutative Algebra, World Science publishing Co.Pte.Ltd.,(1995). 2.2
[8] R. Rasuli, Norms over intuitionistic fuzzy subrings and ideals of a ring, Notes on Intuitionistic Fuzzy Sets, 22(5)(2016), 72-83. 1, 2.11, 2.12
[9] R. Rasuli, t-norms over fuzzy ideals (implicative, positive implicative) of BCK-algebras, Mathematical Analysis and its Contemporary Applications, 4(2)(2022), 17-34. 1
[10] R. Rasuli, T-fuzzy subbigroups and normal T-fuzzy subbigroups of bigroups, J. of Ramannujan Society of Mathematics and Mathematical Sciences, 9(2)(2022), 165-184. 1
[11] R. Rasuli, M. A. Hashemi and B. Taherkhani, S-norms and Anti fuzzy ideals of BCI-algebras, 10th National Mathematics Conference of the Payame Noor University, Shiraz, May, 2022. 1
[12] R. Rasuli, B. Taherkhani and H. Naraghi, T-fuzzy SU-subalgebras, 10th National Mathematics Conference of the Payame Noor University, Shiraz, May, 2022. 1
[13] R. Rasuli, A study of T-fuzzy multigroups and direct preoduct of them, 1th National Conference on Applied Reserches in Basic Sciences(Mathematics, Chemistry and Physics) held by University of Ayatolla Boroujerdi, Iran, during May 26-27, 2022. 1
[14] R. Rasuli, S - (M,N)-fuzzy subgroups, 1th National Conference on Applied Reserches in Basic Sciences(Mathematics, Chemistry and Physics) held by University of Ayatolla Boroujerdi, Iran, during May 26-27, 2022. 1
[15] R. Rasuli, Intuitionistic fuzzy BCI-algebras (implicative ideals, closed implicative ideals, commutative ideals) under norms, Mathematical Analysis and its Contemporary Applications, 4(3)(2022), 17-34. 1
[16] R. Rasuli, Fuzzy d-algebras under t-norms, Eng. Appl. Sci. Lett. (EASL), 5(1)(2022), 27-36. 1
[17] R. Rasuli, Fuzzy vector spaces under norms, Annals of Mathematics and Computer Science, 9(2022), 7-24. 1
[18] R. Rasuli, T-fuzzy G-submodules, Scientia Magna, 17(1)(2022), 107-118. 1
[19] R. Rasuli, Anti Fuzzy Congruence on Product Lattices, 7^{th} International Congerence on Combinatotcs, Crytography, Computer Science and Computing held by Iran University of Science and Technology, Iran,Tehran, during November 16-17, 2022. 1
[20] R. Rasuli, Anti complex fuzzy Lie subalgebras under S-norms, Mathematical Analysis and its Contemporary Applications, 4(4)(2022), 13-26. 1
[21] R. Rasuli, t-conorms over anti fuzzy subgroups on direct product of groupos, Annals of Mathematics and Computer Science, 10(2022), 8-18. 1
[22] A. Rosenfeld, Fuzzy groups, J. Math. Anal. Appl., 35(1971), 512-519. 1
[23] S. Sessa, On fuzzy subgroups and fuzzy ideals under triangular norms: short communiction, Fuzzy Sets and Systems, 13(1984), 95-100. 1
[24] L. A. Zadeh, Fuzzy sets, inform. and Control, 8(1965), 338-353. 1
[25] P. Zhan and Tan, Intuitionistic M-fuzzy subgroups, Soochow Journal of Mathematics, 30(2004), 85-90. 1